Localisation and Balance in an Ensemble Kalman Filter

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Outline

- Illustrate that covariance localisation can produce imbalance.
- Present new localisation formulations.
- Show results from identical-twin experiments in a global shallow-water model, comparing localisation formulations.
- Conclusions.

Covariance Localisation – Why?

$$\mathbf{X}^a = \mathbf{X}^b + \mathbf{P}_e^b\mathbf{H}^T(\mathbf{H}\mathbf{P}_e^b\mathbf{H}^T + \mathbf{R})^{-1}(\mathbf{Y} - \mathbf{H}\mathbf{X}^b)$$

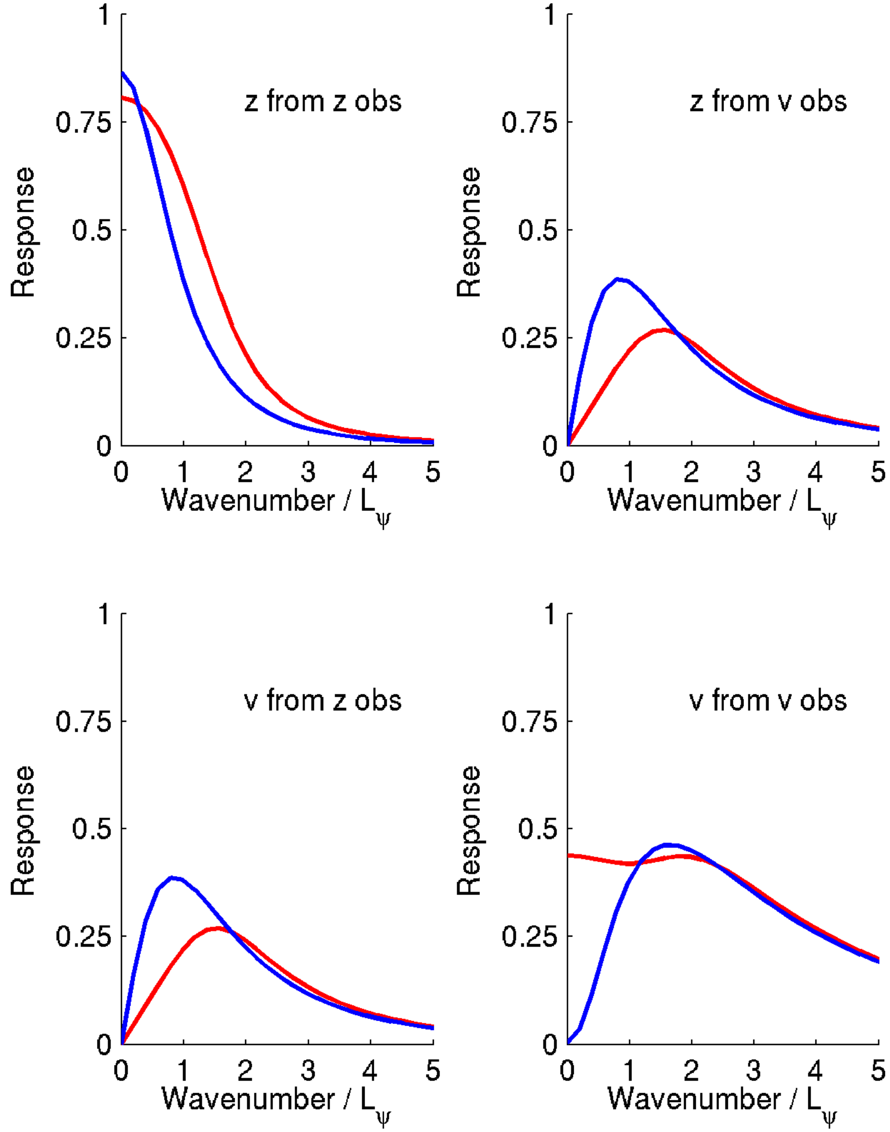
$$\mathbf{X}^a = \mathbf{X}^b + \left[(\mathbf{C}\mathbf{H}^T)\circ(\mathbf{P}_e^b\mathbf{H}^T)\right]\left[(\mathbf{H}\mathbf{C}\mathbf{H}^T)\circ(\mathbf{H}\mathbf{P}_e^b\mathbf{H}^T) + \mathbf{R}\right]^{-1}(\mathbf{Y} - \mathbf{H}\mathbf{X}^b)$$

- $\mathbf{P}_e^b$ is estimated by a Monte Carlo process, so is subject to sampling error.
- Localisation
  - Removes spurious long-range correlations
  - Corrects erroneous response of analysis mean and variance to distant observations.
  - $\mathbf{C}\circ\mathbf{P}_e^b$ is still a covariance (Gaspari and Cohn, 1999)
- rank($\mathbf{P}_e^b$) = O(100) is too small, gives poor fit of analysis to observations
  - Localisation increases rank of $\mathbf{P}_e^b$
- Eliminates analysis discontinuities between sub-domains.

Disadvantages of Localisation

- Weakens balances encoded in $\mathbf{P}_e^b$.
  - Exact balances correspond to the null space of $\mathbf{P}_e^b$. (Lorenc 2003)
  - Advantage of EnKF advantage is better covariances - localisation reduces this.
  - Canadian experience confirms balance problems. (Houtkamer et al 2005)
- Makes analysis suboptimal.
  - Changes the gain matrix.
- *ad hoc* procedure which requires tuning.

Localisation and Balance



- One-dimensional cyclic domain.
- Very dense observations.
- Bivariate (*z*,*v*) analysis using OI covariance model with exact geostrophic balance, nondivergent.
- Plotted response is from diagonals of Fourier transformed gain matrix. (c.f. Daley 1991)

- Blue: global analysis, standard OI covariance model.
- Red: localised covariance model (support = 6*L*).

Improved Localisation

Write the velocity components in terms of streamfunction ψ and velocity potential χ:

$$
\begin{aligned}
u &= -\frac{\cos\phi}{a}\frac{\partial\psi}{\partial\phi} + \frac{1}{a}\frac{\partial\chi}{\partial\lambda} \\
v &= \frac{1}{a}\frac{\partial\psi}{\partial\lambda} + \frac{\cos\phi}{a}\frac{\partial\chi}{\partial\phi}
\end{aligned} \tag{1}
$$

The two-point covariance $\langle\psi_1, \psi_2\rangle$ is

$$
F_{\psi,\psi} = F_{\psi,\psi}(\lambda_1, \phi_1, \lambda_2, \phi_2) = \langle\psi(\lambda_1, \phi_1), \psi(\lambda_2, \phi_2)\rangle = \langle\psi_1, \psi_2\rangle \tag{2}
$$

and similarly for $F_{\psi,\chi}$, $F_{\chi,\psi}$ and $F_{\chi,\chi}$. Then the two-point covariance $F_{u,u} = \langle u_1, u_2\rangle$ becomes

$$
\begin{aligned}
F_{u,u} = {} & \frac{\cos\phi_1\cos\phi_2}{a^2}\frac{\partial}{\partial\phi_1}\frac{\partial}{\partial\phi_2}F_{\psi,\psi} - \frac{\cos\phi_1}{a^2}\frac{\partial}{\partial\phi_1}\frac{\partial}{\partial\lambda_2}F_{\psi,\chi} \\
& - \frac{\cos\phi_2}{a^2}\frac{\partial}{\partial\lambda_1}\frac{\partial}{\partial\phi_2}F_{\chi,\psi} + \frac{1}{a^2}\frac{\partial}{\partial\lambda_1}\frac{\partial}{\partial\lambda_2}F_{\chi,\chi}
\end{aligned} \tag{3}
$$

Improved Localisation (cont'd)

Let C be a localisation function, and apply the localisation be replacing $F_{\psi,\psi} \leftarrow CF_{\psi,\psi}$ in (3) and similarly for $F_{\psi,\chi}$, $F_{\chi,\psi}$ and $F_{\chi,\chi}$. Then the *localised* two-point covariance $\langle u, u \rangle_L$ becomes

$$\begin{aligned} \langle u_1, u_2 \rangle_L = {} & C_{o,o} F_{u,u} \\ & + C_{x,o} F_{\chi,u} - C_{y,o} F_{\psi,u} + C_{o,x} F_{u,\chi} - C_{o,y} F_{u,\psi} \\ & + C_{x,x} F_{\chi,\chi} - C_{x,y} F_{\chi,\psi} - C_{y,x} F_{\psi,\chi} + C_{y,y} F_{\psi,\psi} \end{aligned} \tag{4}$$

The first term on the RHS is the usual localisation. The F's can all be calculated from the ensemble in the usual way. (The EnKF will need to read in ψ and χ in addition to u and v.)

Improved Localisation (cont'd)

The C's are spatial derivatives of the localiser:

$$
\begin{aligned}
C_{o,o} &= C = C(r(\lambda_1, \phi_1, \lambda_2, \phi_2)) \\
C_{x,o} &= \frac{1}{a}\frac{\partial C}{\partial \lambda_1} \\
C_{y,o} &= \frac{\cos\phi_1}{a}\frac{\partial C}{\partial \phi_1} \\
C_{o,x} &= \frac{1}{a}\frac{\partial C}{\partial \lambda_2} \\
C_{o,y} &= \frac{\cos\phi_2}{a}\frac{\partial C}{\partial \phi_2} \\
C_{x,x} &= \frac{1}{a^2}\frac{\partial^2 C}{\partial \lambda_1 \partial \lambda_2} \\
C_{x,y} &= \frac{\cos\phi_2}{a^2}\frac{\partial^2 C}{\partial \lambda_1 \partial \phi_2} \\
C_{y,x} &= \frac{\cos\phi_1}{a^2}\frac{\partial^2 C}{\partial \phi_1 \partial \lambda_2} \\
C_{y,y} &= \frac{\cos\phi_1\phi_2}{a^2}\frac{\partial^2 C}{\partial \phi_1 \partial \phi_2}
\end{aligned}
\qquad (5)
$$

Improved Localisation (cont'd)

Similarly, the localised versions of the other covariances are

$$\begin{aligned}\langle u_1, v_2\rangle_L = {} & C_{o,o}F_{u,v} \\ & + C_{x,o}F_{\chi,v} - C_{y,o}F_{\psi,v} + C_{o,x}F_{u,\psi} + C_{o,y}F_{u,\chi} \\ & - C_{x,x}F_{\psi,\psi} + C_{x,y}F_{\chi,\chi} - C_{y,x}F_{\psi,\psi} + C_{y,y}F_{\chi,\psi}\end{aligned} \tag{6}$$

$$\begin{aligned}\langle v_1, u_2\rangle_L = {} & C_{o,o}F_{v,u} \\ & + C_{x,o}F_{\psi,u} + C_{y,o}F_{\chi,u} + C_{o,x}F_{v,\chi} - C_{o,y}F_{v,\psi} \\ & + C_{x,x}F_{\psi,\chi} - C_{x,y}F_{\psi,\psi} + C_{y,x}F_{\chi,\chi} - C_{y,y}F_{\chi,\psi}\end{aligned} \tag{7}$$

$$\begin{aligned}\langle v_1, v_2\rangle_L = {} & C_{o,o}F_{v,v} \\ & + C_{x,o}F_{\psi,v} + C_{y,o}F_{\chi,v} + C_{o,x}F_{v,\psi} + C_{o,y}F_{v,\chi} \\ & + C_{x,x}F_{\psi,\psi} + C_{x,y}F_{\psi,\chi} + C_{y,x}F_{\chi,\psi} + C_{y,y}F_{\chi,\chi}\end{aligned} \tag{8}$$

$$\langle u_1, z_2\rangle_L = C_{o,o}F_{u,z} + C_{x,o}F_{\chi,z} - C_{y,o}F_{\psi,z} \tag{9}$$

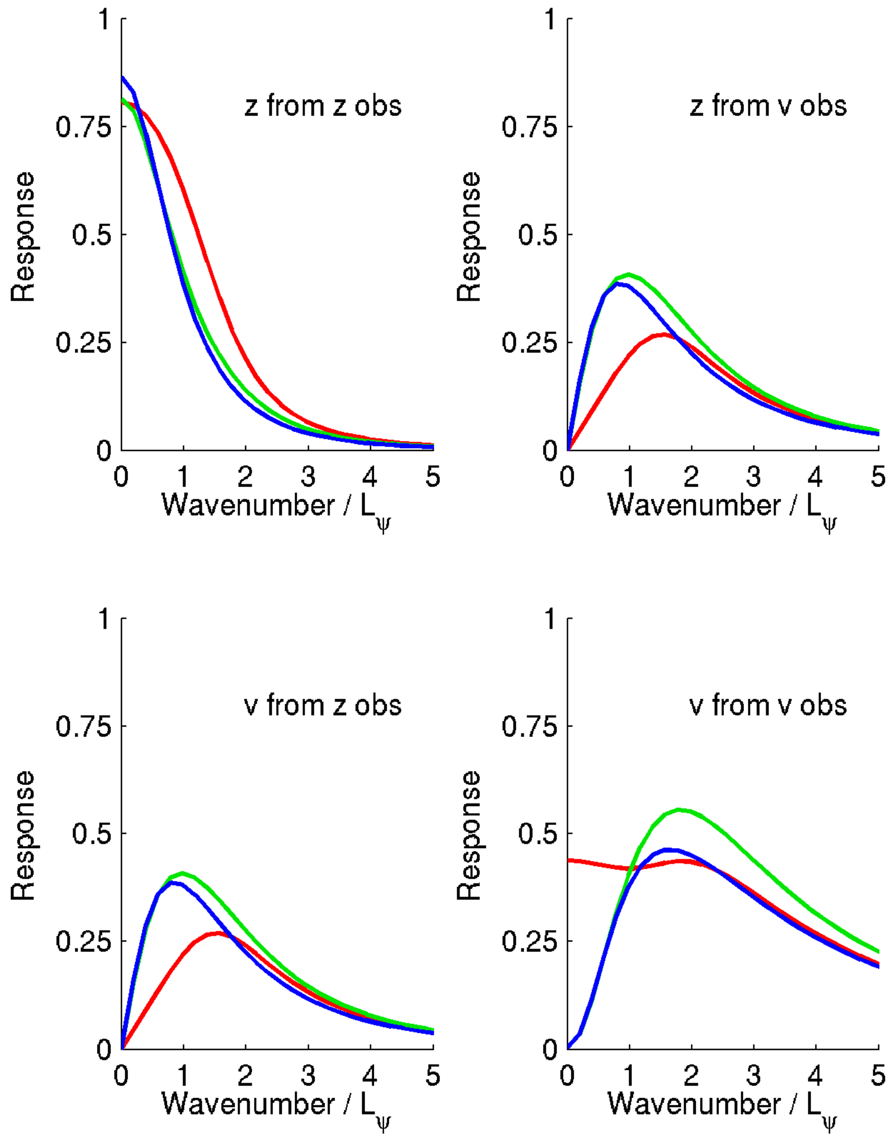
$$\langle v_1, z_2\rangle_L = C_{o,o}F_{v,z} + C_{x,o}F_{\psi,z} + C_{y,o}F_{\chi,z} \tag{10}$$

$$\langle z_1, u_2\rangle_L = C_{o,o}F_{z,u} + C_{o,x}F_{z,\chi} - C_{o,y}F_{z,\psi} \tag{11}$$

$$\langle z_1, v_2\rangle_L = C_{o,o}F_{z,v} + C_{o,x}F_{z,\psi} + C_{o,y}F_{\psi,z} \tag{12}$$

$$\langle z_1, z_2\rangle_L = C_{o,o}F_{z,z} \tag{13}$$

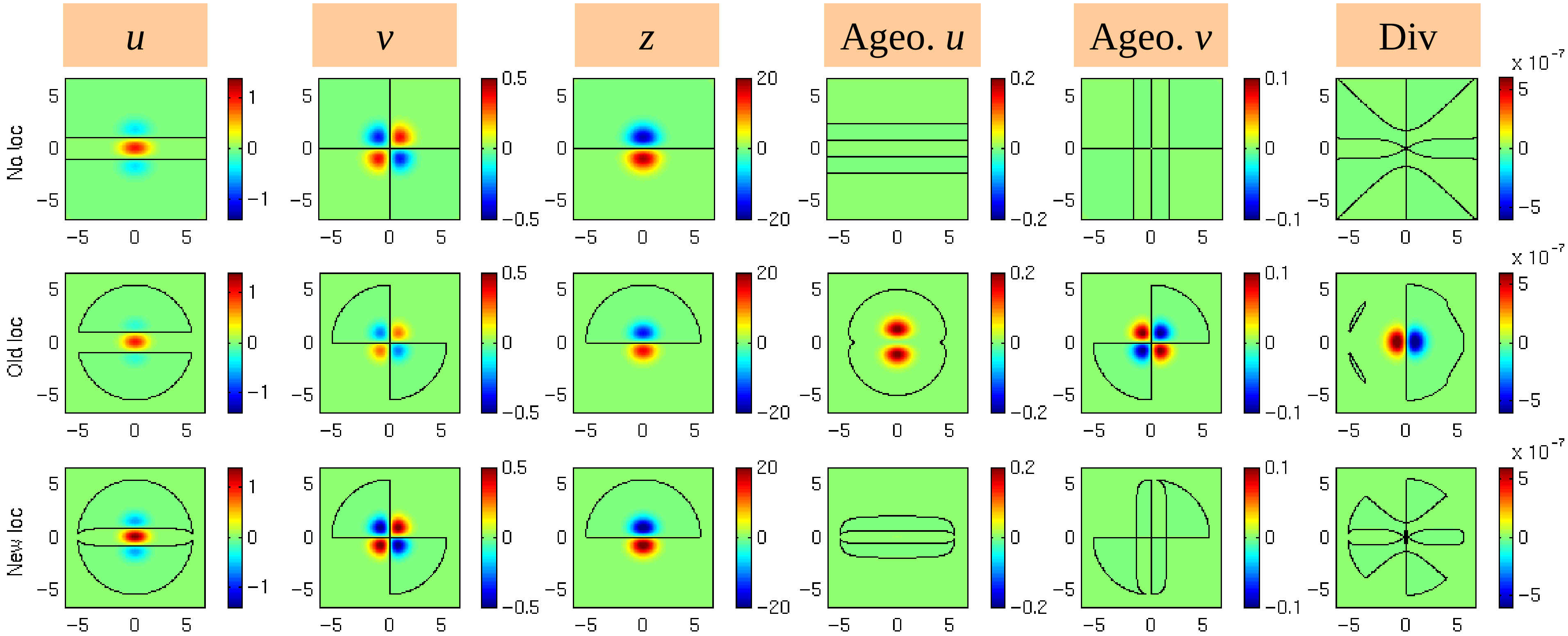
Analysis response using new localisation



- Blue: standard OI covariance model.
- Red: localised covariance model.
- **Green: covariances localised using new method.**

- **Analysis response is geostrophically balanced and nondivergent.**

Improved Localisation Example



- Analysis of single *u* observation, using idealised nondivergent and geostrophically balanced covariances.
- **Top**: No localisation. **Middle**: Standard localisation. **Bottom**: New localisation.

Localise χ cross-covariances to zero

- Experience shows that z-χ and ψ-χ correlation often nearly zero (except in boundary layer).
- It is therefore sensitive to sampling error.
- Can repeat derivation with the χ cross-covariances "localised" to zero.
- Slightly more complex expressions, EnKF now needs to read in rotational and divergent wind components.

A cheap hybrid

- About half of the additional terms in the new formulation can be included in the observation-space to model-space covariances by changing model-space from (z,u,v) to (z,ψ,χ).

$$\mathbf{X}^a = \mathbf{X}^b + \left[(\mathbf{CH}^T)\circ(\mathbf{P}_e^b\mathbf{H}^T)\right]\left[(\mathbf{HCH}^T)\circ(\mathbf{HP}_e^b\mathbf{H}^T)+\mathbf{R}\right]^{-1}(\mathbf{Y}-\mathbf{HX}^b)$$

$$\mathbf{HP}_e^b\mathbf{H}^T = \frac{1}{N-1}\left[\mathcal{H}(\mathbf{X}^b)-\overline{\mathcal{H}(\mathbf{X}^b)}\right]\left[\mathcal{H}(\mathbf{X}^b)-\overline{\mathcal{H}(\mathbf{X}^b)}\right]^T$$

$$\mathbf{P}_e^b\mathbf{H}^T = \frac{1}{N-1}\left[\mathbf{X}^b-\overline{\mathbf{X}^b}\right]\left[\mathcal{H}(\mathbf{X}^b)-\overline{\mathcal{H}(\mathbf{X}^b)}\right]^T$$

model space deviations (z,ψ,χ)

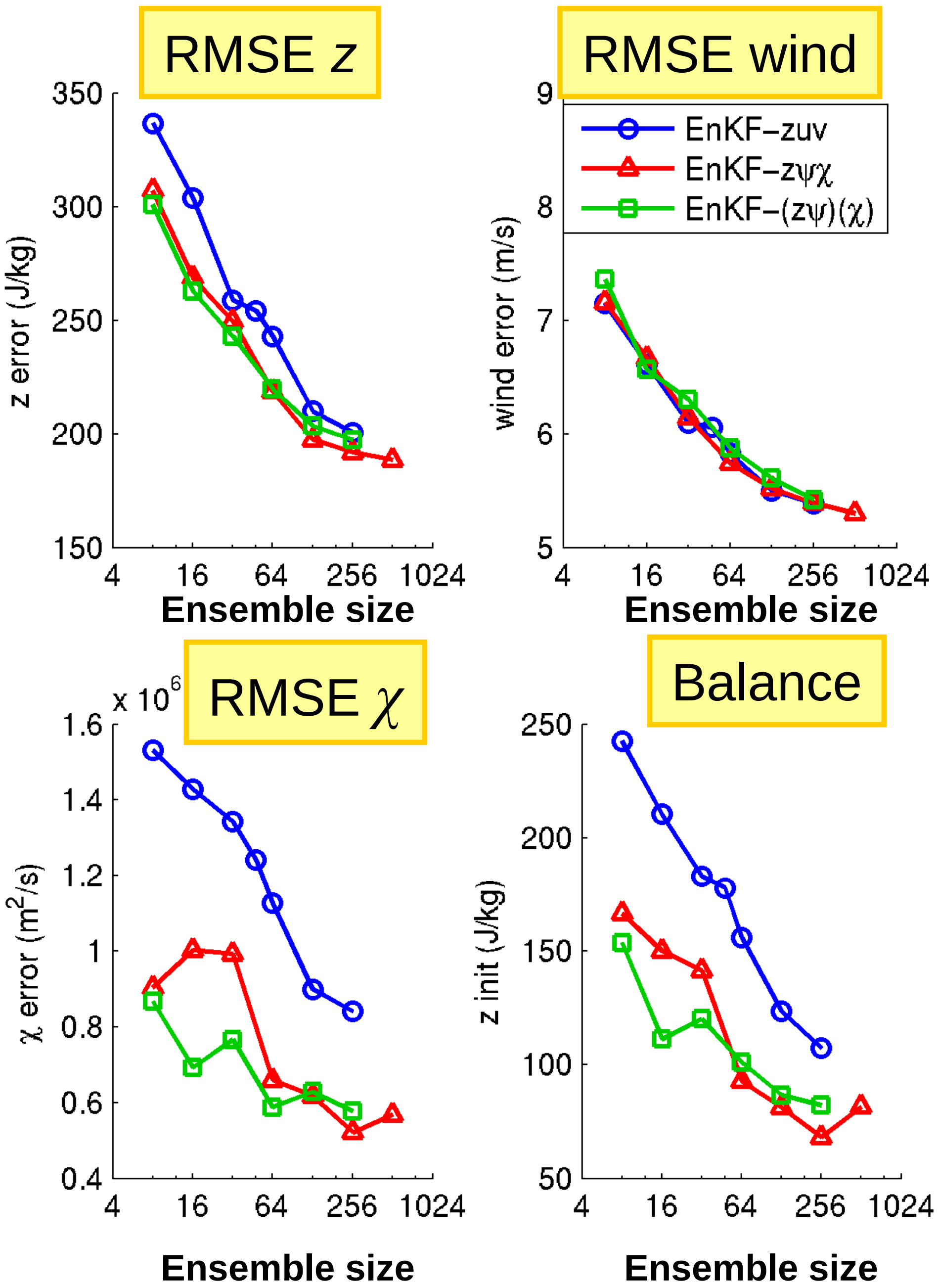
obs space deviations (z,u,v)

Experimental Design

- Global spectral shallow water of Bourke (1972) at T31.
  - Accurate, non-dissipative, includes variable transforms (u,v) to/from (ψ,χ).
  - Simplest model to have main atmospheric balances.
  - Error-doubling time ~ 5 days.
- Identical twin with 60-day truth run
  - Start from ERA-40 analysis for 500 hPa 18 Jan 1962.
  - Initial ensemble from a climate forecast beginning one year before.
  - Analyse last 20 days of assimilation cycle.
- Obs network based on 50% of global radiosondes, observe (z,u,v) 12-hourly.
  - Height error = 100 J/kg, wind error = 5 m/s.
- Standard perturbed-obs EnKF with covariance inflation, various localisations, various ensemble sizes.
  - Tune localisation length and covariance inflation to minimise errors.
  - Verify spread using rank histograms.

Four localisations

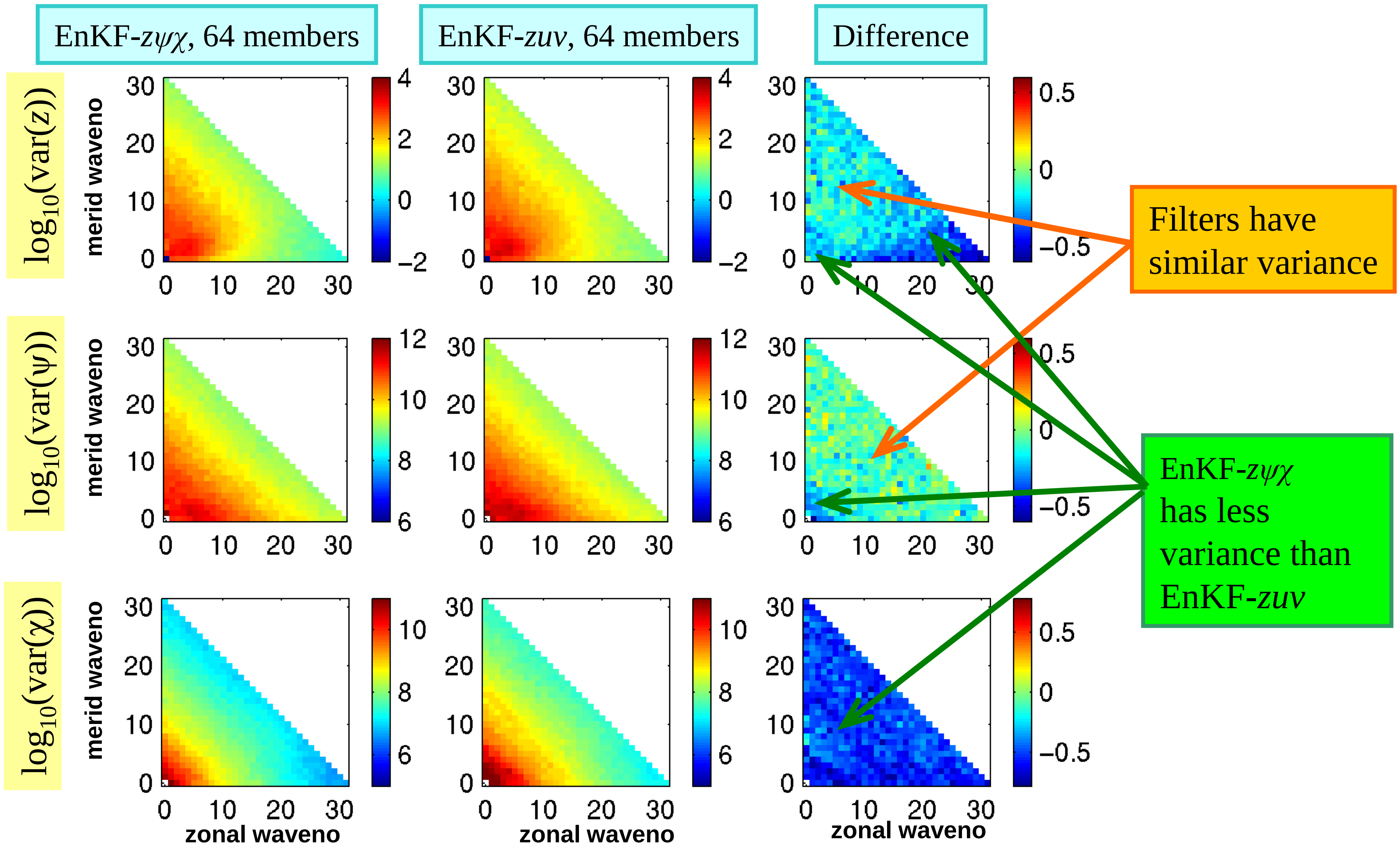
- EnKF-*zuv* : standard localisation.
- EnKF-$z\psi\chi$: new localisation in (z,ψ,χ) space.
- EnKF-hybrid : partial implementation.
- EnKF-$(z\psi)(\chi)$: zeroes out $z-\chi$ and $\psi-\chi$ correlations.



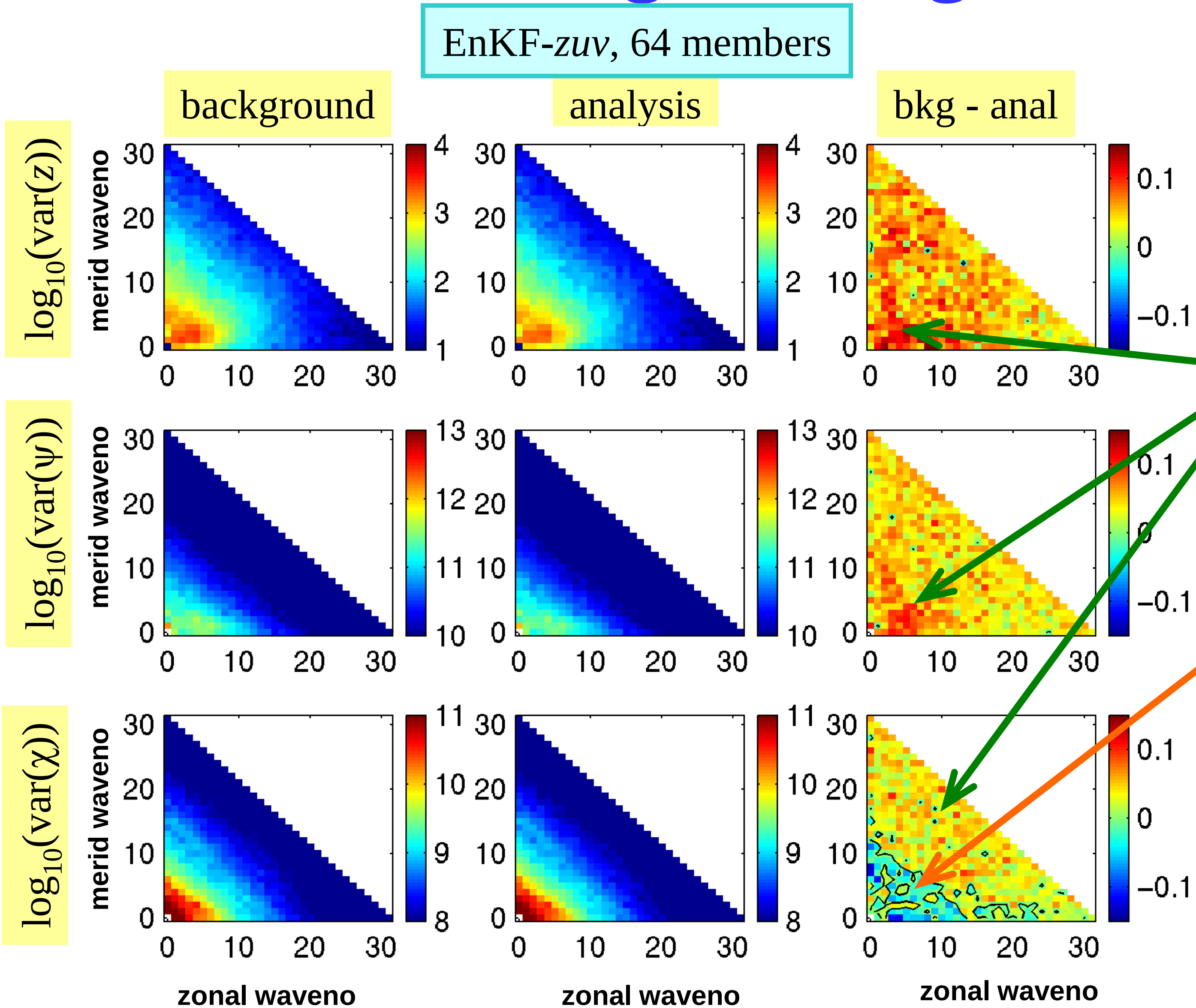
Performance

- Scores improve with bigger ensemble.
  - Better estimation of errors.
- New localisations better than old for height.
  - Equivalent to 50-100% bigger ensemble.
  - Similar for wind.
  - Much better for χ.
- New localisations better balanced.
- EnKF-(zψ)(χ) best for small ensembles, EnKF-zψχ best for large.
- Hybrid is intermediate (not shown).

Background variance in spectral space



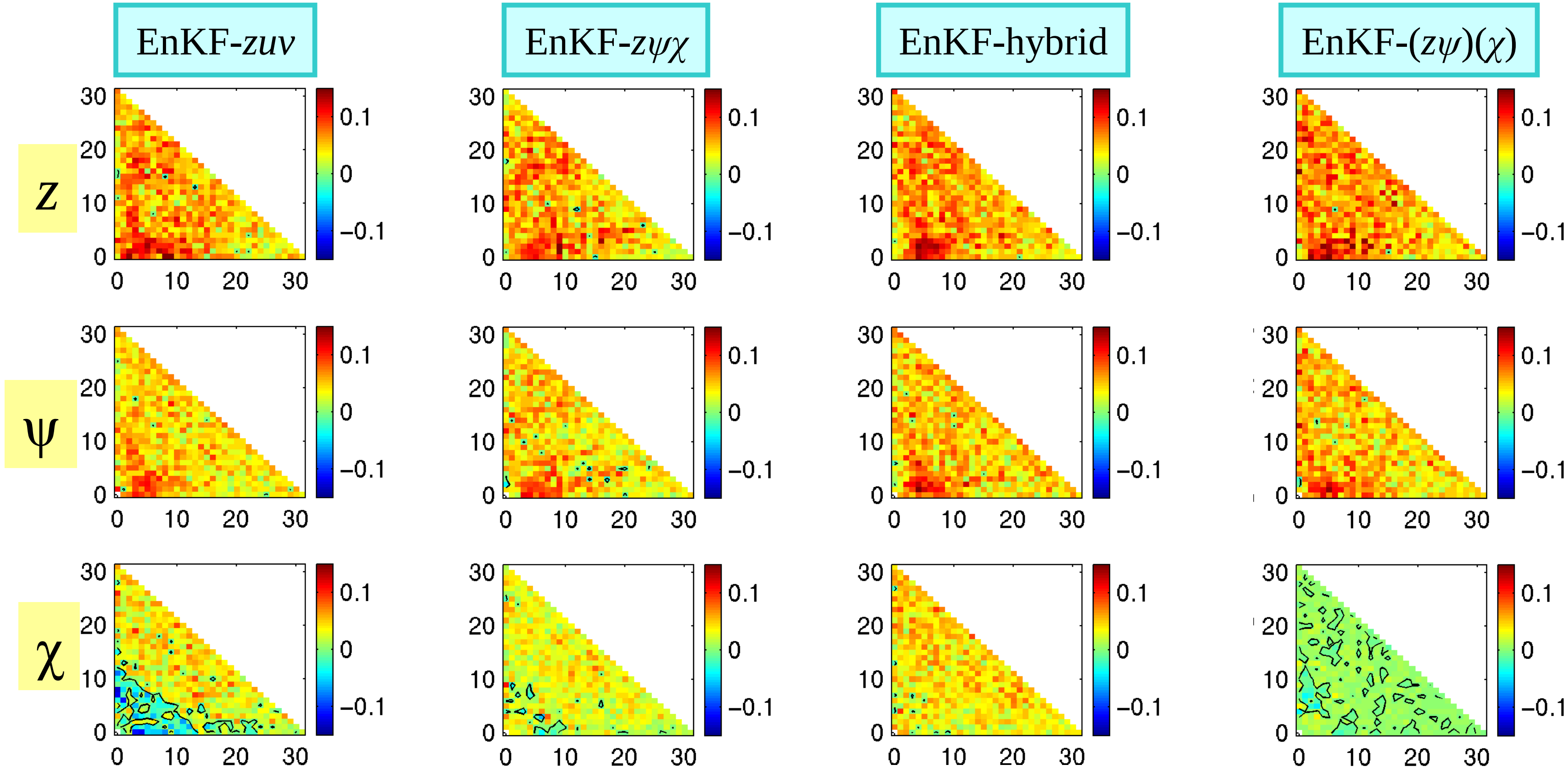
Variance change, background - analysis



Analysis step reduces variance

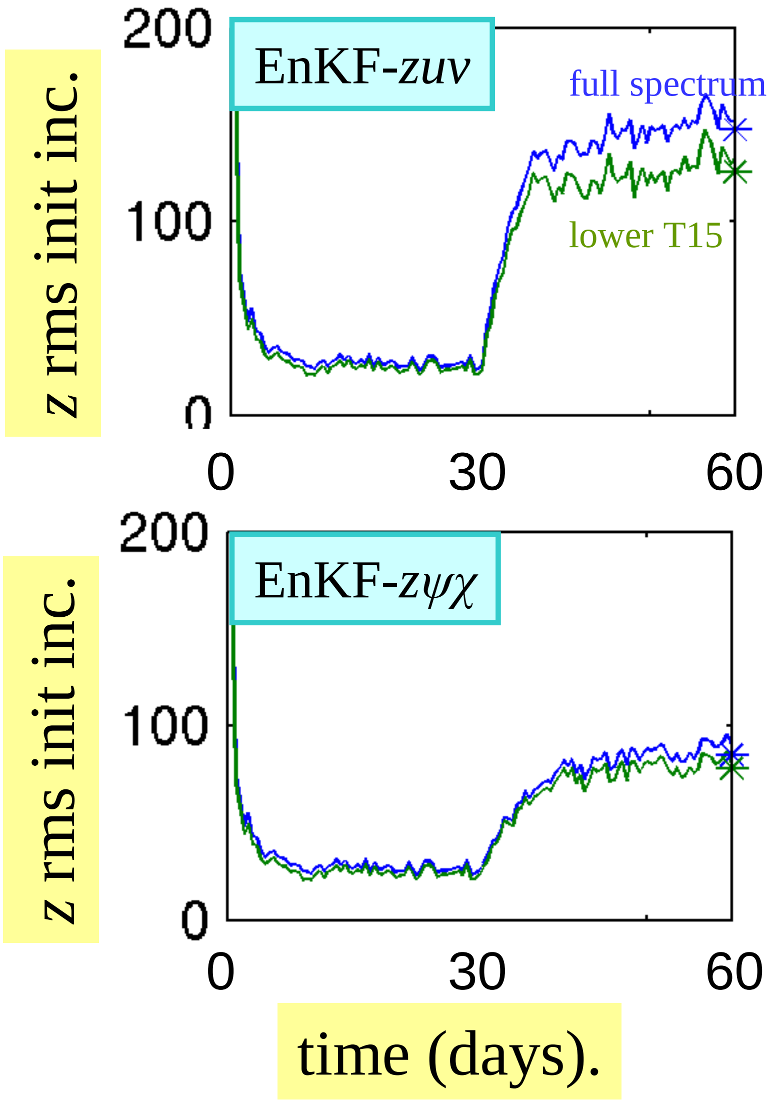
Analysis step increases variance in large-scale divergence.
Suboptimal gain matrix + covariance inflation.

Variance change, background - analysis



- New localisations give
  - Reduced evidence of suboptimal analysis.
  - Somewhat less analysis impact on χ variance.
- This diagnostic includes impact of covariance inflation.

Balance



- Diagnostic is rms(analysis – initialised analysis).
- Switched normal-mode initialisation off in filter at day 30.
- Both filters asymptote to new level of imbalance.
- EnKF-*zuv* has faster imbalance growth, to higher level, with more small-scale imbalance.
- Positive feedback: The analysis will project more of the observations onto the modes with the most variance.
- Similar results with other balance diagnostics (e.g. rms tendencies).

Conclusions

- New localisation is better (produce more accurate and better balanced analyses).
- Removing χ ensemble cross-covariances a further improvement for small ensembles.
- Hybrid realises some of the gain of full scheme.
- Disadvantages of new localisations:
  - Marginally inferior when analysis cycle includes initialisation.
  - Due to additional sampling error (more ensemble covariances calculated)?
  - Extra cost (small except in highly sequential schemes).
  - Increases response to wind observations (can be compensated for).
- Positive feedback on unbalanced modes in EnKF.
- Tested in perturbed-observation-single-EnKF, but applicable to other flavours (except ETKF).